# Practical Tractability of CSPs by Higher-Level Consistency and Tree Decomposition

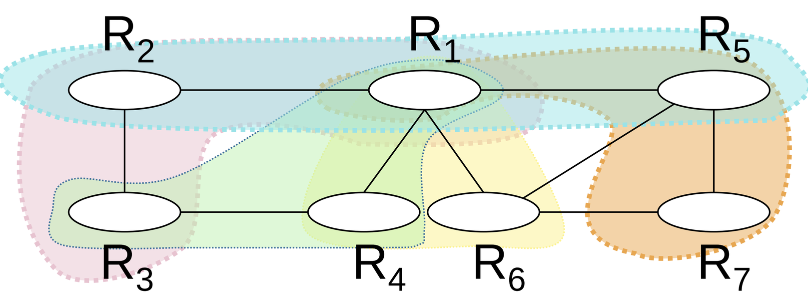
Shant Karakashian, Robert Woodward, and Berthe Y. Choueiry
Constraint Systems Laboratory • University of Nebraska-Lincoln
{shantk|rwoodwar|choueiry}@cse.unl.edu

## Contributions

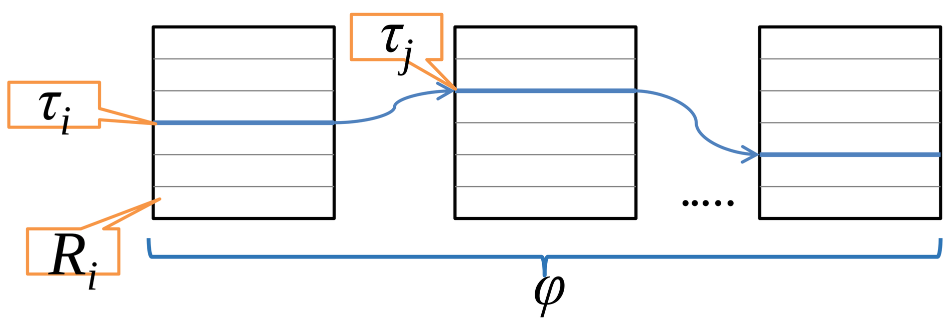
1. A new relational consistency property R(*,*m*)C that does not change the topology of the constraint graph [Karakshian+, AAAI 2010]
2. Two algorithms for enforcing R(*,*m*)C
3. Localizing R(*,*m*)C by restricting it to the clusters of a tree decomposition
4. Bolstering propagation between clusters by adding new constraints at the separators
5. Empirical evidence of practical tractability on benchmark problems

## Relational Consistency

**R(*,*m*)** ensures that subproblem induced in the dual CSP by every connected combination of *m* relations is minimal [Karakshian+, AAAI 2010]



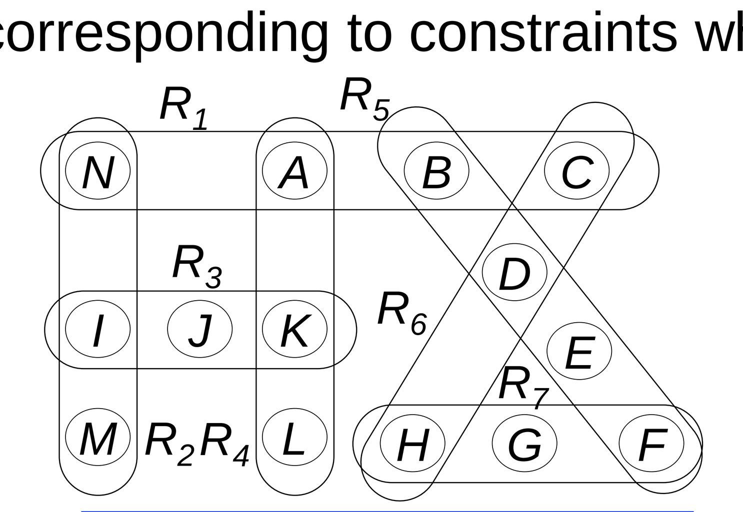

- Number of combinations = $O(e^m)$
- Size of each combination = *m*
- Twelve combinations for R(*,3)C



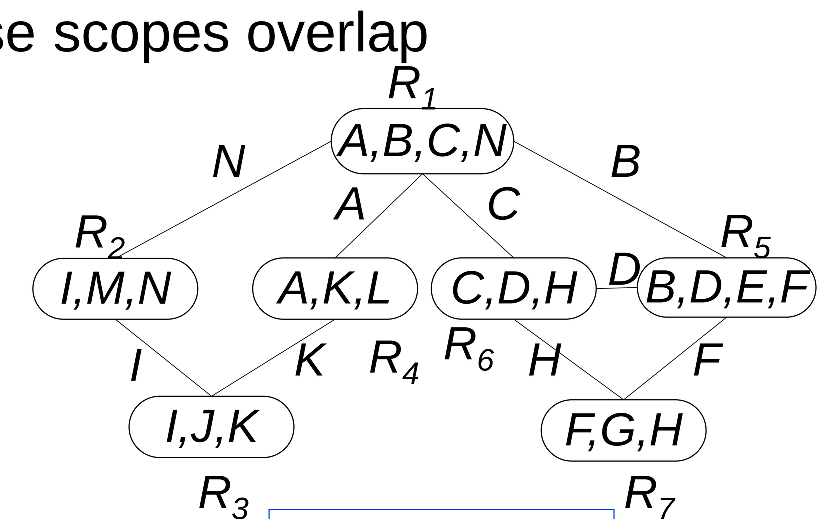

| | |
|---|---|
| 1. $\{R_1,R_2,R_3\}$ | 7. $\{R_1,R_4,R_6\}$ |
| 2. $\{R_1,R_2,R_4\}$ | 8. $\{R_1,R_5,R_6\}$ |
| 3. $\{R_1,R_2,R_5\}$ | 9. $\{R_1,R_5,R_7\}$ |
| 4. $\{R_1,R_2,R_6\}$ | 10. $\{R_1,R_6,R_7\}$ |
| 5. $\{R_1,R_3,R_4\}$ | 11. $\{R_2,R_3,R_4\}$ |
| 6. $\{R_1,R_4,R_5\}$ | 12. $\{R_5,R_6,R_7\}$ |

## Tree Decomposition

The **dual graph** of a CSP is a graph whose vertices represent the constraints of the CSP, and whose edges connect two vertices corresponding to constraints whose scopes overlap

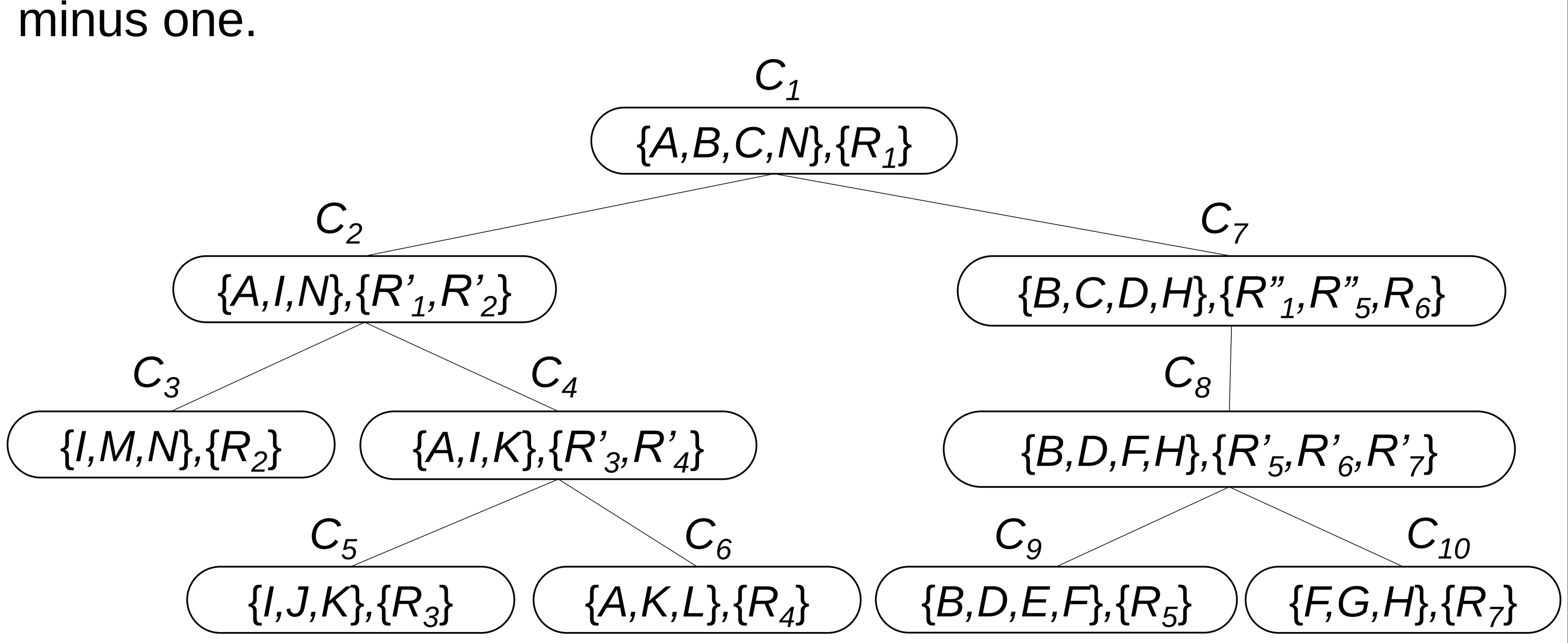



Constraint hypergraph

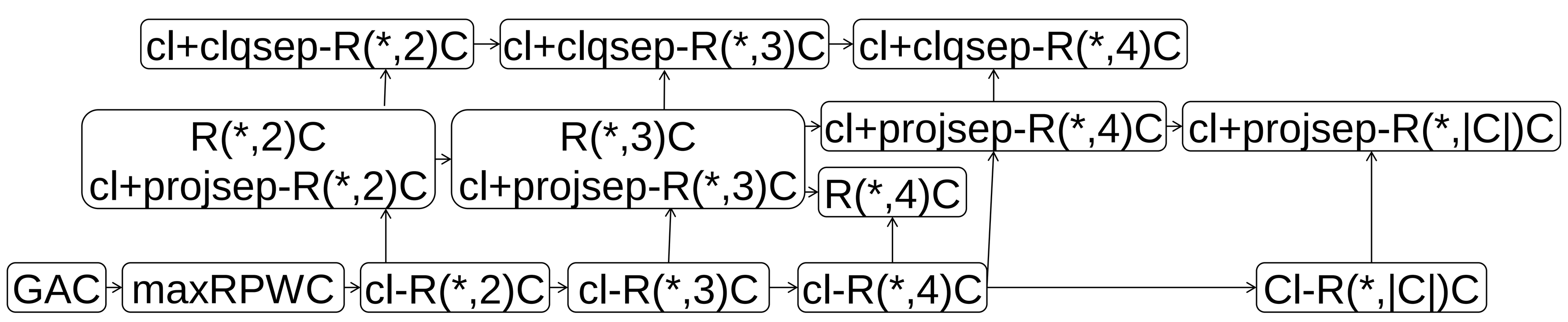



Dual graph

A **tree decomposition** of a CSP is a tree embedding of the constraint network of the CSP. The tree nodes are thus clusters of variables and constraints. A tree decomposition must satisfy two conditions: (1) each constraint appears in at least one cluster and the variables in its scope must appear in this cluster, and (2) for every variable, the clusters where the variable appears induce a connected subtree. A separator of two adjacent clusters is the set of variables in both clusters. A given tree decomposition is characterized by its treewidth, which is the maximum number of variables in a cluster minus one.




## Comparing Consistency Properties



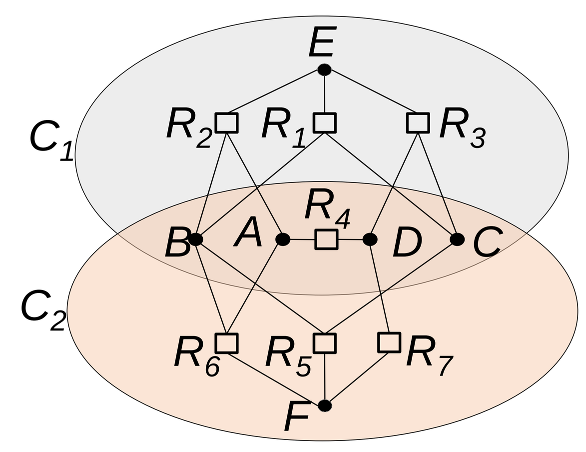

## Bolstering Propagation at Seperators

### Localizing R(*,*m*)C

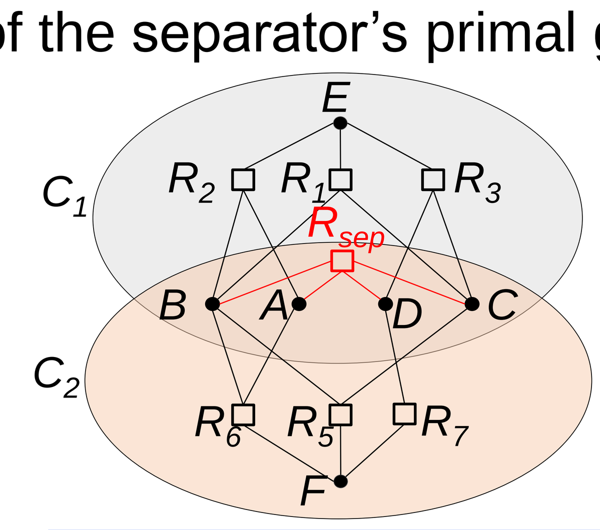
Instead of computing the combinations of *m* constraints over the entire CSP, we restrict ourselves to the combinations computed within each cluster, thus reducing the number of combinations to be considered.

### Bolstering Separators

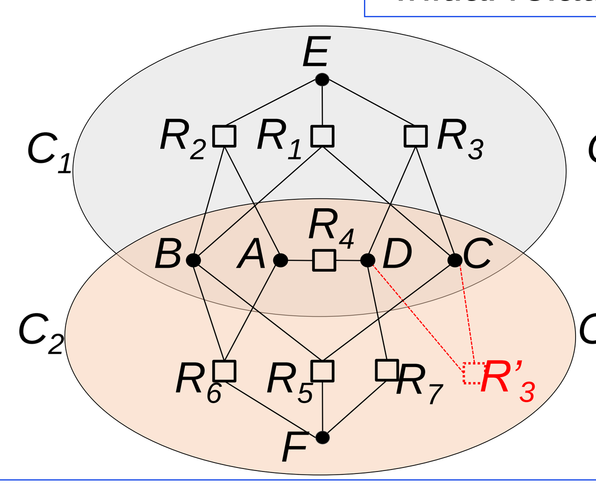
Because the localized R(*,m)C does not consider combinations of relations across clusters, propagation between clusters is hindered. Synthesizing a global constraint at each separator improves the 'communication' between clusters and guarantees backtrack-free search. Synthesizing & storing those global constraints is typically prohibitive, especially for space. We approximate the global constraints by adding redundant constraints at the separators. We propose the following strategies: (1) Adding clusters' constraints to the separator by projecting them on separator's variables; (2) Adding binary constraints to the separator for every fill-in edge obtained by a triangulation of the separator's primal graph; and (3) Adding non-binary constraints to the separator that cover the maximal cliques of a triangulation of the separator's primal graph.



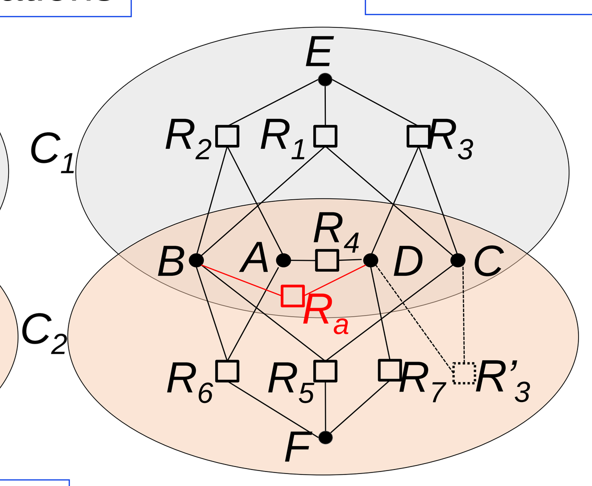

Initial relations



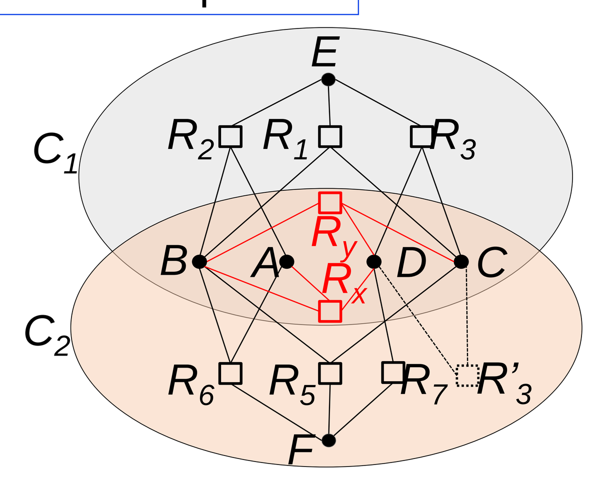

Global relation at separator




Projecting clusters' relations on separator's variables




Binary relations




Max cliques relations

## Empirical Results

- Localized R(*,*m*)C results in weaker filtering but faster consistency algorithm which is useful where the full power of R(*,m)C is not needed.
- Bolstering separators with projected relations results in stronger consistency, solving many more instances in a backtrack-free manner.
- Bolstering separators with binary relations did not reduce #NV.
- Bolstering separators with maximal clique relations further strengthened filtering and solved yet more instances backtrack-free. However, processing the new non-binary relations increased cost. Consequently, many problems could not be solved within 2 hours.

| Total number of instances is 1131 | GAC | maxRPWC | wR(*,2)C | cl-wR(*,2)C | cl+clqsep-wR(*,2)C | wR(*,3)C | cl-wR(*,3)C | cl+projsep-wR(*,3)C | cl+clqsep-wR(*,3)C | wR(*,4)C | cl-wR(*,4)C | cl+projsep-wR(*,4)C | cl+clqsep-wR(*,4)C | Cl-R(*,\|C\|)C | cl+projsep-R(*,\|C\|)C |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| **Comp.** | 296 | 267 | 333 | 288 | 227 | **335** | 334 | **356** | 201 | **336** | 303 | 330 | 189 | **347** | **349** |
| **Fastest** | **125** | **45** | 43 | **44** | 3 | 17 | **46** | 11 | 2 | 15 | 10 | 3 | 3 | **84** | 26 |
| **BTF** | 70 | 111 | 158 | 89 | 160 | 193 | 147 | **235** | 161 | **235** | 147 | **245** | 159 | **214** | **306** |

## Future Research Directions

- Identify problem parameters to select the appropriate consistency for each problem or even for each cluster in a decomposition.
- Validate the approach in the context of solution counting.

Constraint Systems Laboratory

Nebraska Lincoln

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